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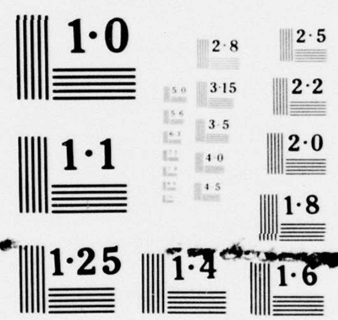

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MULTIVARIATE DISTRIBUTIONS HAVING WEIBULL PROPERTIES*

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ABSTRACT

minimum over i of (a sub i)(X sub i)

Random variables $X_1, \ldots, X_n$ are said to have a joint distribution with
sub 1 sub n
Weibull minimums after arbitrary scaling if $\min_i(a_i X_i)$ has a one dimensional Weibull distribution for arbitrary constants $a_i > 0$, $i = 1, \ldots, n$. Some
a sub i
properties of this class are demonstrated, and some examples are given which show the existence of a number of distributions belonging to the class. One of the properties is found to be useful for computing component reliability importance. The class is seen to contain an absolutely continuous Weibull distribution which can be generated from independent uniform and gamma distributions.

Key words and phrases: multivariate Weibull distributions, Weibull minimums after arbitrary scaling, hazard gradient, component reliability importance, Gumbel's distribution.

AMS (1970) subject classification numbers 62H05 and 62N05

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1. INTRODUCTION

In the following $\overline{F}(\underset{\sim}{x}) = P(X_1>x_1, \ldots, X_n>x_n)$ is the survival function of non-negative random variables $X_1, \ldots, X_n$ and $R = -\log \overline{F}$ is the hazard function which is non-decreasing and defined for non-negative $\underset{\sim}{x}$.

The Weibull distribution, $\overline{F}(x) = \exp(-kx^{\alpha})$, $x\geq 0$, has become an important, often used, model for lifelength. Several multivariate extensions have been suggested ([7], [8], [11]). However, the extensions appear to have little in common with the univariate Weibull distribution except that the marginal distributions are Weibull. An exception is the Weibull distribuion mentioned by Marshall and Olkin [11], and also discussed in [9], which has the following form:

$$\overline{F}(\underset{\sim}{x}) = \exp(-\sum_{J} \lambda_J \max_{i\in J}(x_i^{\alpha})), \quad \underset{\sim}{x}\geq 0 \tag{1.1}$$

with $\alpha>0$ and $\lambda_J>0$ for $J\in\mathcal{J}$ where the sets J are elements of the class $\mathcal{J}$ of non-empty subsets of $\{1, \ldots, n\}$ having the property that for each i, $i\in\mathcal{J}$ for some $J\in\mathcal{J}$. For $\alpha=1$, (1.1) is the Marshall-Olkin [11] multivariate exponential distribution.

The purpose of this paper is to develop some properties of the class of multivariate distributions having Weibull minimums after arbitrary scaling. Random variables $X_1, \ldots, X_n$ have such a distribution if for arbitrary constants $a_i>0$, $i=1, \ldots, n$, $\min_i(a_iX_i)$ has a one dimensional Weibull distribution,

$$P(\min_i(a_iX_i)>t) = \exp(-k(\underset{\sim}{a})t^{\alpha}), \quad t\geq 0 \tag{1.2}$$

for some $\alpha>0$ and constant $k(\underset{\sim}{a})>0$. The Weibull distribution (1.1) belongs to this class, as do a number of other distributions which are presented in the next section.

A closely related subclass of distributions satisfying (1.2) are distributions having exponential minimums after arbitrary scaling. Esary and Marshall [5] discuss this class and other classes of exponential distributions, and their application to computing system reliability.

The following section contains examples and comparisons of various classes of Weibull distributions which show that distributions satisfying (1.2) are distinct from other classes of Weibull distributions. In later sections failure rate, dependence, and distributional properties of this class are presented. A useful application is made to computing the reliability importance of system components. The final section is concerned with generating an absolutely continuous Weibull distribution (which satisfies (1.2)) from independent random variables, and the effect of the parameters on the covariance.

2. CLASSES OF WEIBULL DISTRIBUTIONS

To clarify differences between distributions satisfying (1.2) and other classes of multivariate Weibull distributions it is helpful to consider a hierarchy of classes of multivariate Weibull distributions.

Consider random variables $X_1, \ldots, X_n$ having a joint distribution which satisfies one of the following conditions.

(a) $X_1, \ldots, X_n$ are independent and each X_i has a Weibull distribution of the form $\overline{F}_i(t) = \exp(-\lambda_i t^\alpha)$, $t \geq 0$, $i=1, \ldots, n$.

(b) $X_1, \ldots, X_n$ have a multivariate Weibull distribution generated from independent Weibull distributions by letting

$$X_i = \min(Z_J\colon i \in J),\ i=1, \ldots, n,$$

where the sets J are elements of a class $\mathcal{J}$ of nonempty subsets of $\{1, \ldots, n\}$ having the property that for each i, $i \in J$ for some $J \in \mathcal{J}$, and the random variables Z_J, $J \in \mathcal{J}$, are independent having Weibull distributions of the form $\overline{F}_J(t) = \exp(-\lambda_J t^\alpha)$.

(c) $X_1, \ldots, X_n$ have a joint distribution satisfying (1.2).

(d) $X_1, \ldots, X_n$ have a joint distribution with Weibull minimums, that is,

$$P(\min_{i \in S}(X_i) > t) = \exp(-\lambda_S t^\alpha)$$

for some $\lambda_S > 0$ and all nonempty subsets S of $\{1, \ldots, n\}$.

(e) Each X_i, $i=1, \ldots, n$ has a Weibull distribution of the form $\overline{F}_i(t) = \exp(-\lambda_i t^\alpha)$.

The classes a-e contain the corresponding classes of multivariate exponential distributions constructed by Esary and Marshall [5]. Each class satisfies certain multivariate closure properties similar to those that they describe. See their properties P_1, P_2, P_3 and P_4. Also each class a-e is a subclass of the one which follows it.

The condition (b) is an alternative and equivalent way to describe the distributions of (1.1). The representation of (1.1) in terms of independent random variables is discussed in [9].

The examples which follow show the classes a-e are distinct since each class is seen to contain distributions not belonging to the class preceeding it.

Example 2.1. Let $\overline{F}(x_1,x_2) = \exp[-(\lambda_1 x_1^\alpha + \lambda_2 x_2^\alpha + \lambda_{12} \max(x_1^\alpha, x_2^\alpha)]$ with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} \geq 0$ and $\alpha > 0$. This is the bivariate version of the multivariate distributions satisfying (b). This distribution occurs if $X_1 = \min(Z_1, Z_{12})$ and $X_2 = \min(Z_2, Z_{12})$ where Z_1, Z_2, Z_{12} are independent with Weibull distributions $P(Z_1 > t) = \exp(-\lambda_1 t^\alpha)$, $P(Z_2 > t) = \exp(-\lambda_2 t^\alpha)$, $P(Z_{12} > t) = \exp(-\lambda_{12} t^\alpha)$. If $\lambda_{12} > 0$ then the joint distribution of X_1, X_2 satisfies (b) but not (a).

Example 2.2. Let X_1, X_2 have the joint distribution of example 2.1 and let $Y_i = c_i^{-1} X_i$, with $c_i > 0$, $i=1,2$. Then $\overline{F}(y_1,y_2) = [-(\lambda_1 c_1^\alpha y_1^\alpha + \lambda_2 c_2^\alpha y_2^\alpha$

$+ \lambda_{12} \max(c_1^\alpha y_1^\alpha, c_2^\alpha y_2^\alpha)]$. The distribution of Y_1, Y_2 has a singular component on the line $c_1\, y_1 = c_2\, y_2$. Thus it differs from the distributions satisfying (b). If $c_1 \neq c_2$ the joint distribution of Y_1 and Y_2 satisfies (c) but not (b).

Example 2.3. $\bar{G}(x_1,x_2) = \exp[-(x_1^4+x_2^4)^{\frac{1}{2}}]$ satisfies (c) but not (b). In a later section it is shown that this distribution can be generated by a transformation of independent random variables. $\bar{G}(x_1,x_2)$ is absolutely continuous and therefore cannot satisfy (b). That it satisfies (c) can be verified by computing $P(\min_i(a_i X_i) > t) = \exp[-t^2(a_1^{-4} + a_2^{-4})^{\frac{1}{2}}]$, $t \geq 0$, for $a_i > 0$, $i=1,2$.

Example 2.4. Let $\bar{H}(x_1,x_2) = \bar{G}(x_1,x_2)\bar{F}(x_1,x_2)$ where $\bar{F}$ is the distribution of example 2.1 with $\alpha=2$ and $\bar{G}(x_1,x_2)$ is the distribution of example 2.3. $\bar{H}(x_1,x_2)$ is not absolutely continuous and satisfies (c) but not (b).

Example 2.5. Let X_1,X_2 have the distribution $\bar{G}(x_1,x_2)$ of example (2.3) and let Y_1,Y_2 have the distribution $\bar{F}(y_1,y_2) = \exp[-(2y_1^8 + 2y_2^8)^{\frac{1}{4}}]$. Let $(T_1,T_2) = (X_1,X_2)$ with probability p and $(T_1,T_2) = (Y_1,Y_2)$ with probability 1-p. Then T_1,T_2 have the distribution of the mixture $\bar{H}(t_1,t_2) = p\,\bar{G}(t_1,t_2) + (1-p)\bar{F}(t_1,t_2)$. The distribution $\bar{H}$ satisfies (d) but not (c).

Example 2.6. Let $\bar{F}(x_1,x_2) = \bar{F}_1(x_1)\bar{F}_2(x_2)[1 + \gamma(1-\bar{F}_1(x_1))(1-\bar{F}_2(x_2))]$ where $\bar{F}_j(x_j) = \exp(-x_j^{c_j})$, $c_j > 0$, $x_j \geq 0$, $j=1,2$ are univariate Weibull distributions. This bivariate Weibull distribution is mentioned in [7] as a special case of the Morgenstern, Gumbel, Farlie distributions. It satisfies (e) when $c_1=c_2$, but does not satisfy (d).

In summary, the class of Weibull distributions (1.2) contains independent Weibull distributions satisfying (a) and the class of Weibull distributions (b) arising from the Marshall-Olkin [11] models. Examples (2.2), (2.3) and (2.4) show the existence of other Weibull distributions satisfying (1.2) which are distinct from the classes (a) and (b).

3. PROPERTIES OF DISTRIBUTIONS HAVING WEIBULL MINIMUMS AFTER ARBITRARY SCALING

In the present section it is assumed that $\bar{F}(\underset{\sim}{x}) = \exp(-R(\underset{\sim}{x}))$ is a continuous function of $\underset{\sim}{x}$, but not necessarily absolutely continuous, and that the hazard gradient, $r_j(\underset{\sim}{x}) = \frac{\partial}{\partial x_j} R(\underset{\sim}{x})$, $j=1, \ldots, n$ exists except possibly on a finite set of values of x_j. Further, it is assumed that $r_j(\underset{\sim}{x})$ is a continuous function of x_j with the exception of the points where it fails to exist.

It follows that the survival function can be recovered by integrating in the following way,

$$\int_{x_j}^{\infty} r_j(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n)\bar{F}(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n)dt \tag{3.1}$$

$$= \bar{F}(\underset{\sim}{x}).$$

The equality holds irrespective of the way $r_j(\underset{\sim}{x})$ is defined at its discontinuity points. However, to express the results of this section we let $r_j(\underset{\sim}{x})$ represent the right hand derivative which is assumed to exist for all $\underset{\sim}{x}$.

Absolutely continuous distributions satisfy such conditions as do also the multivariate Weibull distributions satisfying (1.1). For the distribution (1.1), $\bar{F}(\underset{\sim}{x})$ is a continuous function of $\underset{\sim}{x}$ and $r_j(\underset{\sim}{x}) = \sum_J \lambda_J \alpha x_j^{\alpha-1} I_J(\underset{\sim}{x})$ where $I_J(\underset{\sim}{x}) = 1$ (and zero otherwise) if $j \in J$ and $x_j > \max\{x_i : i \in J \text{ and } i \neq j\}$. It is seen that $r_j(\underset{\sim}{x})$ is continuous in x_j except on a finite set of values and can be defined at the exceptional values by the right hand derivative.

The hazard gradient is useful for describing failure rate properties of multivariate distributions. In [7] it is shown that $r_j(\underset{\sim}{x})$ can be interpreted as the failure rate of the conditional distributions of X_j given that $X_i > x_i$, $i \neq j$, $i=1, \ldots, n$. It reduces to the usual concept of failure rate when the distribution involves independent random variables. Further discussion of the hazard gradient is given in [10].

A distribution $\overline{F}$ satisfies (1.2) if and only if the hazard function satisfies the following functional equation:

$$R(t\underset{\sim}{x}) = t^{\alpha}R(\underset{\sim}{x}) \text{ for some } \alpha>0 \text{ whenever } t\geq 0 \text{ and } \underset{\sim}{x}\geq 0. \tag{3.2}$$

Equation (3.2) is the basis in this section for developing properties of distributions having Weibull minimums after arbitrary scaling.

Theorem 1. Let $X_1, \ldots, X_n$ have a joint distribution satisfying (1.2) with $\alpha>0$ given by (1.2) and having the hazard gradient $r_j(\underset{\sim}{x})$, $j=1, \ldots, n$. Then

a. $r_j(t\underset{\sim}{x}) = t^{\alpha-1}r_j(\underset{\sim}{x})$, $j=1, \ldots, n$ for all vectors $\underset{\sim}{x}\geq 0$ and scalar $t\geq 0$.

b. $r_j(\underset{\sim}{x})$ is nonincreasing in x_i for $i\neq j$, $i=1, \ldots, n$.

c. $r_j(\underset{\sim}{x})$ is nondecreasing in x_j, $j=1, \ldots, n$ providing $\alpha\geq 1$.

Proof: a. Using (3.2) write $R(\underset{\sim}{x}) = x_j^{\alpha}R(1j, x_j^{-1}\underset{\sim}{x})$ where the notation $(1j, x_j^{-1}\underset{\sim}{x})$ represents a vector with a one in the jth position and the remaining elements have been multiplied by the scalar x_j^{-1}. For $i\neq j$, $r_i(\underset{\sim}{x}) = x_j^{\alpha} \frac{\partial}{\partial x_i} R(1j,x_j^{-1}\underset{\sim}{x})$ $= x_j^{\alpha-1}r_i(1j,x_j^{-1}\underset{\sim}{x})$. Therefore, $r_i(t\underset{\sim}{x}) = (tx_j)^{\alpha-1}r_i(1j,x_j^{-1}\underset{\sim}{x}) = t^{\alpha-1}r_i(\underset{\sim}{x})$, for any $\underset{\sim}{x}\geq 0$ and $t\geq 0$.

b. First observe from (3.2) that $\frac{-\partial}{\partial x_j}\overline{F}(t\underset{\sim}{x}) = t^{\alpha}r_j(\underset{\sim}{x})\overline{F}(t\underset{\sim}{x})$ for $t>0$. Since $-t^{-\alpha}\frac{\partial}{\partial x_j}\overline{F}(t\underset{\sim}{x})$ is non-increasing in x_i, for $i\neq j$, and all $t>0$, and since $\lim_{t\to 0^+} -t^{-\alpha}\frac{\partial}{\partial x_j}F(t\underset{\sim}{x}) = r_j(\underset{\sim}{x})$, we have that $r_j(\underset{\sim}{x})$ is non-increasing in x_i for $i\neq j$.

c. From part a, $r_j(\underset{\sim}{x}) = x_j^{\alpha-1}r_j(1j,x_j^{-1}\underset{\sim}{x})$. Also from part b, $r_j(1j,x_j^{-1}\underset{\sim}{x})$ is non-decreasing in x_j, and since by assumption $\alpha\geq 1$, it follows that $r_j(\underset{\sim}{x})$ is non-decreasing in x_j.

As pointed out in references [2] and [4] a form of positive dependence is likely to be a reasonable assumption for many reliability problems. For random variables $X_1, \ldots, X_n$ satisfying (1.2), part b of the theorem can be used to show that each subset S of the variables is right tail increasing (See [4] for a discussion of right tail increasing) in the remaining set $\bar{S}$. That is, the conditional probability

$$P(X_i > x_i,\ i \in S \mid X_j > y_j,\ j \in \bar{S}) = \exp[-R(\underset{\sim}{x},\underset{\sim}{y}) + R(\underset{\sim}{0},\underset{\sim}{y})]$$

is nondecreasing in y_j, $j \in \bar{S}$. From part b we have $\frac{\partial}{\partial y_j} R(\underset{\sim}{x},\underset{\sim}{y})$ is nonincreasing in x_i. Therefore, $\frac{\partial}{\partial y_j} R(\underset{\sim}{x},\underset{\sim}{y}) \leq \frac{\partial}{\partial y_j} R(\underset{\sim}{0},\underset{\sim}{y})$, which says that $R(\underset{\sim}{x},\underset{\sim}{y}) - R(\underset{\sim}{0},\underset{\sim}{y})$ is nonincreasing in y_j, $j \in \bar{S}$. This proves right tail increasing for distributions (1.2).

For a second application consider $X_1, \ldots, X_n$ satisfying (1.2) with $\alpha \geq 1$. This corresponds to $\min_i(a_i X_i)$ having a one dimensional IFR (increasing failure rate) Weibull distribution for each choice of constants $a_i > 0$, $i=1, \ldots, n$. Part c of the theorem shows that the distributions (1.2) have the property that Johnson and Kotz [7] call multivariate IHR (increasing hazard rate).

Next consider $V = \min_i(X_i)$ and define the event that X_j coincides with V by

$$X_j = V \iff X_j \leq \min_{i \neq j}(X_i). \tag{3.3}$$

Since for distributions satisfying (b) of section 2 there is positive probability of tied values, it is important to note when computing $P(X_j = V)$ that equality is allowed in (3.3).

To develop a special property of distributions satisfying (1.2) let us write

$$P(X_j = V \text{ and } V > x) = \int_x^\infty P(\min_{i \neq j}(X_i) \geq t \mid X_j = t) f_j(t)\,dt \tag{3.4}$$

since the density $f_j(t)$ of X_j exists for distributions (1.2).

The integrand in (3.4) is equal to

$$\lim_{\Delta\to 0^+} \Delta^{-1} P(\min_{i\neq j}(X_i)\geq t \text{ and } t\leq X_j<t+\Delta)$$

$$= r_j(t, \ldots, t)\overline{F}(t, \ldots, t). \tag{3.5}$$

The integrand is also equal to

$$P(X_j=V|V=t)g(t) \tag{3.6}$$

where $g(t) = -\frac{d}{dt}\overline{F}(t, \ldots, t)$ is the density function of V. Equating (3.5) and (3.6) gives the conditional probability,

$$P(X_j=V|V=t)=r_j(t, \ldots, t)\overline{F}(t, \ldots, t)[g(t)]^{-1} \tag{3.7}$$

The following theorem extends a property of the Marshall-Olkin [11] distribution (see [2]) to the class of distributions having Weibull minimums after arbitrary scaling.

Theorem 2. Let $X_1, \ldots, X_n$ have a joint distribution satisfying (1.2) with hazard gradient $r_j(\underset{\sim}{x})$ computed as the right hand derivative. Then V is independent of the events $X_j=V$, $j=1, \ldots, n$ and $P(X_j=V) = r_j(1, \ldots, 1)/\alpha R(1, \ldots, 1)$.

Proof: Since for distributions satisfying (1.2), $g(t) = \alpha t^{\alpha-1}R(1, \ldots, 1)\overline{F}(t, \ldots, t)$, and from theorem 1, part a, $r_j(t, \ldots, t) = t^{\alpha-1}r_j(1, \ldots, 1)$ it is seen that (3.7) simplifies to $r_j(1, \ldots, 1)/\alpha R(1, \ldots, 1)$. Therefore, $P(X_j=V|V=t)$ is constant in t which proves the independence of V and $X_j=V$.

4. APPLICATION - COMPUTING COMPONENT RELIABILITY IMPORTANCE.

Let $\tau(\underset{\sim}{X})$ represent the life length of a coherent system having minimal path sets $P_1, \ldots, P_p$ and suppose $X_1, X_2, \ldots, X_n$ represent component life lengths. Then $\tau(\underset{\sim}{X}) = \max_{j=1,\ldots,p}(\tau_j)$ where $\tau_j = \min_{m\in P_j}(X_m)$, $j=1, \ldots, p$. This representation of system life length in terms of minimal path sets is discussed in [2].

Barlow and Proschan [3] define their measure of a component's reliability importance as the probability that component lifelength coincides with system life length. If this event occurs the component is said to cause the system to fail. Since

$$P(X_i = \tau(\underset{\sim}{X})) = P(\max_{j=1,\dots,p} (\tau_j) = X_i)$$

is the probability of the union of p events, the importance measure can be expressed as follows:

$$P(X_i = \tau(\underset{\sim}{X})) = \sum_{j=1}^{p} P(\tau_j = X_i) - \sum_{j,k=1;j\neq k}^{p} P(\min(\tau_j, \tau_k) = X_i)$$

$$+ \dots \pm P(\min(\tau_1, \dots, \tau_p) = X_i). \qquad (4.1)$$

Barlow and Proschan [3] express their formulas for the importance measure in terms of the system reliability function for the case of independent component life lengths, and do not mention (4.1).

Noting that $\min(\tau_j,\tau_k) = \min_{m\in P_j \cup P_k} (X_m)$, and so on, it is seen that each term of the various sums reduces to computing probabilities like those expressed in theorem 2. Note also that if $i \notin P_j \cup P_k$ and if $X_1, X_2, \dots, X_n$ have an absolutely continuous distribution then $P(\min(\tau_j,\tau_k) = X_i) = P(\min_{m\in P_j \cup P_k} (X_m) = X_i) = 0$. Other terms may equal zero for the same reason.

To illustrate the application of theorem 2 for a two out of three system, let X_1, X_2, X_3 represent component life lengths having the joint distributions $\overline{F}(x) = \exp[-(x_1^2 + 2x_2^2 + 3x_3^2)^{\frac{1}{2}}]$. A two out of three system fails when any two of its components fail. System life length is $\tau(\underset{\sim}{X}) = \max[\min(X_1,X_2), \min(X_2,X_3), \min(X_1,X_3)]$. Using (4.1) it is seen that $P(X_1 = \tau(\underset{\sim}{X})) = P(X_1 = \min(X_1,X_2)) + P(X_1 = \min(X_1,X_3)) - 2P(X_1 = \min(X_1,X_2,X_3))$, since the remaining terms become zero for the reason mentioned above. From theorem 2 we have $P(X_1 = \min(X_1,X_2)) = r_1(1,1,0)/R(1,1,0) = {}^1/_3$, $P(X_1 = \min(X_1,X_3)) = r_1(1,0,1)/R(1,0,1) = {}^1/_4$, and

$P(X_1=\min(X_1,X_2,X_3)) = r_1(1,1,1)/R(1,1,1) = {}^1/_6$. Thus the probability that component #1 causes the system to fail is ${}^1/_3 + {}^1/_4 - {}^1/_3 = {}^1/_4$. Similar computations would show $P(X_2 = \tau(\underset{\sim}{X})) = {}^2/_5$ and $P(X_3 = \tau(\underset{\sim}{X})) = {}^7/_{20}$.

5. AN ABSOLUTELY CONTINUOUS WEIBULL DISTRIBUTION

Consider the following bivariate Weibull distribution:

$$\overline{F}(x_1,x_2) = \exp[-(\lambda_1 x_1^{\beta} + \lambda_2 x_2^{\beta})^{\gamma}] \tag{5.1}$$

with $\lambda_i>0$, $x_i\geq 0$, $i=1,2$, $\beta>0$ and $0<\gamma\leq 1$. This distribution has the properties discussed in section 3. For $\beta\gamma=1$ it reduces to Gumbel's [6] bivariate exponential distribution and has several properties in common with the Marshall-Olkin distribution, e.g., exponential marginals, exponential minimums after arbitrary scaling and the independence property discussed in theorem 2. The distribution easily extends to n variables.

Let us show that random variables X_1,X_2 having distribution (5.1) can be represented in terms of independent random variables. Such a representation can be useful for analyzing properties of the distribution and generating random samples.

Consider the random variables

$$Z_i = \lambda_i X_i^{\beta},\ i=1,2 \tag{5.2}$$

and their joint distribution given by

$$\overline{G}(z_1,z_2) = \exp[-(z_1+z_2)^{\gamma}]. \tag{5.3}$$

The joint density function is of the form

$$g(z_1,z_2) = [\gamma(1-\gamma)(z_1+z_2)^{\gamma-2}+\gamma^2(z_1+z_2)^{2\gamma-2}]\exp[-(z_1+z_2)^{\gamma}]. \tag{5.4}$$

Consider next the transformation

$$U = Z_1/(Z_1+Z_2) \tag{5.5}$$
$$S = (Z_1+Z_2)^{\gamma}$$

having the jacobian $(1/\gamma)S^{\frac{2}{\gamma}-1}$.

The joint density of U and S is given by

$$h(u,s) = [(1-\gamma)+\gamma s]e^{-s} \tag{5.6}$$

$0<u<1$, $0<s<\infty$. Thus U and S are independent random variables with U having a uniform distribution on the interval (0,1) and the distribution of S is a mixture of gamma distributions having the density

$$h(s) = [1-\gamma+\gamma s]e^{-s},\ s>0. \tag{5.7}$$

In summary we have from (5.5) that

$$\begin{aligned} Z_1 &= US^{1/\gamma} \\ Z_2 &= (1-U)S^{1/\gamma} \end{aligned} \tag{5.8}$$

are represented in terms of independent random variables U and S.

It is an easy exercise to compute the covariance from the distributions of U and S:

$$\mathrm{COV}(Z_1,Z_2) = (1/\gamma)\Gamma(2/\gamma)-(1/\gamma^2)\Gamma^2(1/\gamma) \tag{5.9}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function.

The covariance is non-negative. Using formulas 6.1.2 and 6.1.18 for the gamma function given in [1], it is possible to show that the covariance is decreasing in γ so that Z_1 and Z_2 are more associated for γ taking a value near zero. As γ approaches one the covariance becomes zero. For $\gamma=1$, Z_1 and Z_2 are independent random variables.

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